# An Analytical Discussion on Optical Fibers and Ion-Acoustic Modeling

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## Introduction to Optics

Optics is a branch of physics that deals with the study of light and its interactions with matter. It encompasses the behavior of light, its properties, its production, transmission, detection, and manipulation. Optics is crucial in various fields including engineering, medicine and astronomy. New technologies have emerged as a result of optics advancements in recent years including fiber optics for high-speed communication, lasers for a range of industrial and medical uses, and optical computing for possible gains in computer power. Overall, the study of optics is a fascinating field that contributes to our understanding of the nature of light and has broad applications in many branches of science and technology. Its further research and development should open up new avenues for innovation in the years to come.

## History and Significance of Optical Fibers

The history of optical fibers begins in the early nineteenth century when researchers began experimenting with the idea of transmitting light through glass fibers. Scottish engineer John Logie Baird identified the practical application of transmitting images through bundles of optical fibers in the early 1920s. These fibers were not suitable for long-distance communication due to their high stress.

Researchers began experimenting with optical fibers for medical, communication, and sensing applications in the 1950s and 1960s. In 1951, Indian-American physicist Narinder Singh Kapani coined the term "fiber optics" and is called the "Father of Fiber Optics." In 1954, physicists Harold Hopkins and Narinder Singh Kapani published their article in the British journal "Nature." In that article, they proposed sending images through bundled glass fiber which in turn paves the way for the utilization of optical fibers for communication purposes. Early in the 1960s, a team of American researchers at Corning Glass Works (now Corning Inc.) led by Robert Mauer, Donald Keck, and Peter Schultz made substantial progress toward producing high-quality glass fibers. Because of the low attenuation rates of the fibers, light can travel over long distances without experiencing a discernible drop in signal intensity.

In a study published in 1966, researchers Charles Kao and George Hockham published a paper proposing the use of glass fiber for telecommunications. They have described the potential of low-loss optical fibers made of silica glass which lays the foundation for modern optical communications. In the 1980s, the global deployment of optical fiber networks increased dramatically, propelled by the growing demand for high-speed, long-distance communications. The capacity and efficiency of optical communication systems were further increased by the introduction of fiber-optic amplifiers and wavelength-division multiplexing (WDM).

The practical development of optical fiber began in the early twentieth century. However, major progress in the creation of practical optical fiber technology was not achieved until the middle of the 20th century. The deployment of optical fiber networks increased dramatically in the late 20th

century as a result of the growing need for high-speed communications infrastructure. Because fiber-optic cables offer far higher bandwidth and reduced signal loss than conventional copper cables, they have replaced the latter in long-distance communications networks.

In the twenty-first century, new materials and techniques have allowed for even more breakthroughs in optical fiber technology including longer distances and quicker data transmission. These days, optical fiber serves as the backbone of modern telecommunications networks supporting the internet, television, telephone, and other communications services worldwide. The usage of optical fiber in a multitude of industries, such as industrial automation (data transfer in factories), sensing (environmental monitoring), and medicine (endoscopy), in addition to telecommunication, further solidifies its significance in modern civilization.

## The Ion-Acoustic Model

The ion-acoustic model, which explains waves in plasmas composed primarily of ions and electrons, is a fundamental concept in plasma physics. Experimental studies of ion-acoustic waves began in the 1960s-1970s. Researchers have conducted laboratory experiments and observations in astrophysical contexts (such as the solar wind) to detect these waves and study their properties. Researchers explored nonlinear effects in ion-acoustic waves, leading to the study of solitons and shock waves within plasma contexts. The ion-acoustic model has also been applied in various fields, such as space physics, fusion research, and biomedical applications. The ion-acoustic model continues to be an active area of research, helping us understand plasma dynamics and wave phenomena in a variety of contexts.

The ion-acoustic model generally describes the behavior of ion-acoustic waves (IAWs) in plasma, which are low-frequency oscillations involving ions and electrons. Electron pressure drives these waves which are crucial for comprehending various plasma phenomena. Ion-acoustic waves occur in plasmas where ions and electrons interact, and generate a variety of wave phenomena. Understanding these models is crucial for space physics, fusion research, and ion-based devices. The study of ion-acoustic models using fractional derivatives helps capture the subtleties of wave behavior over time including dissipative and dispersive effects that are often overlooked in classical models. One of the most important features of ion-acoustic waves in plasmas is their ability to form coherent structures, such as solitons and shock waves that can propagate over long distances without dissipating. These structures arise due to the interplay between the dispersive, diffusive, and nonlinear effects in the plasma, and their properties can be studied using the ZK equation. Solitons are stable and localized wave packets that maintain their shape as they propagate through the plasma on the other hand, while shock waves are characterized by a rapid increase in wave amplitude and are often associated with energy dissipation.

In recent developments, fractional derivatives have been used to generalize the ion-acoustic model, effectively accounting for long-range interactions that are significant in more complex or anomalous plasmas. When fractional derivatives (such as the Beta, Caputo, or Riemann-Liouville derivatives) are introduced into ion-acoustic models, they alter the time evolution and spatial relationships in the governing equations. The use of fractional derivatives provides flexibility in modeling the decay rate of ion-acoustic waves, with fractional orders better fitting experimental data and offering a more accurate description of real-world plasma systems.

## Fractional-Order Nonlinear Evolution Equations (FNLEEs)

Therefore, Some FNLEEs to examine their closed-form traveling wave solutions. The considered equations are the (2+1)-dimensional time-fractional Nizhnik-Novikov-Veselov equation, the space-time fractional nonlinear Schrödinger Kerr law equation, the space-time fractional unstable nonlinear Schrödinger equation, the space-time fractional Schrödinger-Hirota equation, the space-time fractional Biswas-Arshed equation, the space-time fractional Sasa-Satsuma equation, the space-time fractional nonlinear Kaup-Newell equation and the (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov equations and their solutions are crucial for extending classical models to take into account non-local interactions and complex geometrics. They are essential for accurate modeling of waves in a variety of physical systems, including nonlinear optics, quantum mechanics, fluid dynamics, and plasma physics. Thus, in this thesis, we have investigated the above-mentioned models by the double $(\phi'/\phi, 1/\phi)$-expansion method.

The mentioned equations have already been investigated for their analytic and numerical solutions through some approaches.

### (2+1)-Dimensional Time-Fractional Nizhnik-Novikov-Veselov (NNV) Equation

The (2+1)-dimensional time-fractional Nizhnik-Novikov-Veselov equation is represented as:

$$D_t^{\alpha}\psi = a\psi_{xxx} + b\psi_{yyy} - 3a\phi_x\psi - 3a\phi\psi_x - 3b\omega\psi_y - 3b\omega_y\psi$$

$$\psi_x = \phi_y, \psi_y = \omega_x$$

Details relating to each term are given below: $D_t^{\alpha}\psi$ represents the fractional derivative of $\psi$ with respect to time of order $\alpha$. $\psi(x, y, t)$ is the wave variable. $(x, y)$ are spatial coordinates. $t$ is the time. $\alpha$ represents the order of the fractional derivative with respect to time. $a, b$ are constants. $\psi_{xxx}$ and $\psi_{yyy}$ represent third-order spatial derivatives with respect to $x$ and $y$, respectively. $\phi_x$ and $\psi_x$ denote the spatial derivatives of $\phi$ and $\psi$ with respect to $x$. $\omega_y$ and $\psi_y$ denote the spatial derivatives of $\omega$ and $\psi$ with respect to $y$. $\phi(x, y, t)$ and $\omega(x, y, t)$ represents are other dependent variables. $\beta$ is another parameter. The equation for "integer order" includes the time derivative with the highest degree. In this case, the highest order of time derivative is given by the fractional derivative operator $D_t^{\alpha}$. Since the equation involves a fractional derivative with respect to time, the order of the equation is non-integer. It depends on the specific value of $\alpha$ chosen for the fractional derivative. Typically, $\alpha$ is a real number between 0 and 1.

The (2+1)-dimensional time-fractional Nizhnik-Novikov-Veselov equation was numerically and analytically studied by the authors (Zhang et al., 2022; Duarte et al., 2017; Li et al., 2014; Fong, 2015; Williams, 2017; Chen, and Wang, 2020; Zhao, and Zhang, 2023; Abdalla et al., 2018; Wang, and Liu, 2022).

### Space-Time Fractional Perturbed Nonlinear Schrödinger Kerr Law Equation

The space-time fractional perturbed nonlinear Schrödinger Kerr law equation with fractional-order represents a mathematical model used to describe the nonlinear propagation of waves in a medium which account for both spatial and temporal dispersion effects. The equation is provided below.

$$i D_t^{\alpha}\psi + a D_x^{2\alpha}\psi + b D_t^{2\alpha}\psi + cF(\vee\psi ¿^2)\psi = 0.$$

The terms in the equation are as follows: $i$ is the imaginary unit and $D_t^{\alpha}\psi$ has the same meaning as discussed in NNV equation. The term $aD_{x,t}^{2\alpha}\psi$ represents the mixed derivative of order $2\alpha$ with respect to both space and time. It captures the coupling between spatial and temporal variations that influence how the dynamics in one dimension affect the other. The term $bD_x^{2\alpha}\psi$ represents the fractional-order derivative in space of order $2\alpha$ acting on the function $\psi$. It accounts for spatial dissipation, indicating how $\psi$ dissipate in space. $cF\left(\vee\psi¿^2\right)\psi$ represents the nonlinear term, where $F$ is a function of the magnitude squared of the wave function $\psi$. $a, b$, and $c$ are constant coefficients. The space-time fractional perturbed nonlinear Schrödinger-Kerr law equation with fractional-order represents an adaptable mathematical model for studying the nonlinear propagation of waves in complex media. In this equation, $\alpha$ represents the order of the fractional derivatives in time and space. These fractional orders can take real values between 0 and 1, indicating the degree of fractional differentiation. The $\alpha$ value indicates the degree of memory or non-locality in time or space. When $\alpha=1$, the standard derivatives are recovered. For $\alpha<1$, the equation incorporates fractional calculus effects, allowing for the description of phenomena with long memory or non-local behaviors.

The space-time fractional nonlinear Schrödinger-Kerr law equation was investigated for exact analytical solutions by several methods, such as the Modified Riemann-Hilbert method (Zhang et al., 2020), the first integral method (Ilyas et al., 2021), and the homogeneous balance method (Wu and He, 2020). Liu et al. (2010) also studied the same equation through the generalized auxiliary equation method, the extended $(G'/G)$-expansion method (Zhang et al., 2014), and the first integral method (Moosaeia et al., 2011) to explore exact solutions.

## Fractional Space-Time Unstable Nonlinear Schrödinger Equation

The fractional space-time unstable nonlinear Schrödinger equation with fractional-order is provided below:

$$iD_t^{\alpha}\psi+D_x^{2\alpha}\psi+2\lambda\left(\vee\psi¿^2\psi\right)-2\gamma\psi=0.$$

The fractional-order of the above-mentioned equation is discussed below. Here, the term $iD_t^{\alpha}\psi$ have the same meaning as discussed in Schrödinger Kerr law equation. In the context of fractional-order, this simplifies to a standard derivative with respect to time. This term describes how the wave function changes over time, taking into account both linear and nonlinear dynamics. Also the term $D_x^{2\alpha}\psi$ has the same meaning as discussed in Schrödinger Kerr law equation. When $\alpha$ is a fractional, this simplifies to a standard second-order spatial derivative. It describes how the wave function varies spatially considering the curvature and spatial variations in the system. This term $2\lambda\left(\vee\psi¿^2\psi\right)$ represents the nonlinear interaction within the system. It describes how the wave function interacts with itself, with a strength determined by the parameter $\lambda$. The term $¿\psi¿^2$ represents the squared magnitude of the wave function indicating the probability density of finding the particle at a particular point in space. Multiplying this by $\psi$ gives the self-interaction term where the wave function affects itself in a nonlinear manner. The term $-2\gamma\psi$ represents an external potential acting on the system. It is linear in $\psi$ indicating a linear interaction between the wave function and the external potential, with a strength determined by the parameter $\gamma$. This term accounts for the influence of an external potential on the wave function. The integer-order fractional space-time unstable nonlinear Schrödinger equation describes the evolution of a wave function $\psi$ in

both time and space which incorporate nonlinear interactions and potential effects. This equation finds applications in various areas of physics, particularly in the study of quantum mechanics, nonlinear optics, and condensed matter physics. Its solutions can reveal phenomena such as wave packet dynamics, soliton formation, and quantum tunneling.

Exact solitary wave solutions of the space-time fractional unstable nonlinear Schrödinger equation were investigated by using the modified extended direct algebraic method (Lu et al., 2018), the variational iteration method (Wazwaz, 2008), the improved auxiliary equation method (Sarwar et al., 2023), and the extended Jacobi's elliptic expansion functions method . Abdelrahman et al., (2020) applied the sine-cosine method to the space-time fractional unstable nonlinear Schrödinger equation and obtained three types of solutions: hyperbolic function solution, trigonometric function solution, and rational function solution.

## Fractional-Order Disrupted Nonlinear Space-Time Schrödinger-Hirota Equation

The fractional-order disrupting the nonlinear space-time the Schrödinger-Hirota equation is a complex equation that describes the evolution of a wave function $\psi$ in a nonlinear optical medium. The equation is as follows:

$$i D_t^{\alpha}\psi + a D_x^{2\alpha}\psi + b D_t^{2\alpha}\psi + c \vee \psi ¿^2 \psi + i\left(\gamma D_{x,t}^{3\alpha}\psi + \theta \vee \psi ¿^2 D_t^{\alpha}\psi\right) = i\delta D_x^{3\alpha}\psi + i\lambda D_{x,t}^{3\alpha}\left(\vee \psi ¿^2 \psi\right) + i\nu D_t^{3\alpha}\left(\vee \psi ¿^2 \psi\right).$$

The terms $i D_t^{\alpha}\psi$, $a D_{x,t}^{2\alpha}\psi$ and $b D_x^{2\alpha}\psi$ have the same meaning as discussed in Schrödinger Kerr law equation. $c \vee \psi ¿^2 \psi$ describes nonlinear interactions within the wave function $\psi$. The term $¿\psi ¿^2$ represents the intensity of $\psi$, and $c$ governs the strength of the nonlinearity. $\gamma D_{x,t}^{3\alpha}\psi$ represents higher-order spatial dispersion effects with a fractional-order 1 derivative of order $\alpha$. It captures more complex spatial behavior of the wave function $\psi$. $\theta \vee \psi ¿^2 D_t^{\alpha}\psi$ involves a nonlinear interaction term. It represents the interaction between the intensity distribution and the spatial evolution of the wave function $\psi$ with a fractional-order derivative of order $\alpha$. $i\delta D_x^{3\alpha}\psi$, $i\lambda D_{x,t}^{3\alpha}\left(\vee \psi ¿^2 \psi\right)$ and $i\nu D_t^{3\alpha}\left(\vee \psi ¿^2 \psi\right)$ are additional nonlinear terms. They introduce further nonlinear interactions between different components of the wave function $\psi$ with fractional-order derivatives of order $\alpha$. The space-time fractional perturbed nonlinear Schrödinger-Hirota equation's integer-order terms are essential for understanding the system's linear and nonlinear dynamics, including dispersion, nonlocal effects, and the interaction of many physical processes. Comprehending these technical terms is crucial for comprehending the behaviour of the system and forecasting its dynamics in diverse settings.

The space-time fractional Schrödinger-Hirota model was examined for analytic solutions by the modified Jacobi elliptic function method (Sharif, 2023), the tanh method (Biswas et al., 2012), the F-expansion method (Onder et al., 2024), the extended auxiliary equation method (Ray, 2020), and obtained soliton solutions, periodic wave solutions, and rational solutions.

## Space-Time Fractional Biswas-Arshed Equation

The space-time fractional Biswas-Arshed equation with non-integer order derivatives is a nonlinear evolution equation that describes the dynamics of a complex function $\psi$ in both space and time. The equation is given below:

$$i D_t^{\alpha}\psi + a_1 D_x^{2\alpha}\psi + a_2 D_t^{2\alpha}\psi + i\left(b_1 D_{x,t}^{3\alpha}\psi + b_2 D_t^{3\alpha}\psi\right) - i\left[\lambda D_x^{\alpha}\left(\vee \psi ¿^2 \psi\right) + \nu D_t^{\alpha}\left(\vee \psi ¿^2\right)\psi + \theta \vee \psi ¿^2 D_t^{\alpha}\psi\right] = 0,$$

where $a_1, a_2, b_1, b_2, \lambda, v, \theta$ are constants. The terms $i D_t^{\alpha}\psi$, $a_1 D_{x,t}^{2\alpha}\psi$ and $a_2 D_t^{2\alpha}\psi$ have the same meaning as discussed in Schrödinger Kerr law equation. $i\left(b_1 D_{x,t}^{3\alpha}\psi + b_2 D_t^{3\alpha}\psi\right)$: These terms represent higher-order mixed fractional-order derivatives in space and time of order $3\alpha$. They introduce further complexity in the dynamics, accounting for higher-order interactions and non-local effects. $-i[\lambda D_x^{\alpha}(\vee\psi ¿^2\psi) + v D_t^{\alpha}(\vee\psi ¿^2)\psi + \theta\vee\psi ¿^2 D_t^{\alpha}\psi]$: These terms represent nonlinear interactions involving the function $\psi$ and its modulus squared $¿\psi¿^2$. They account for self-interaction and nonlinear effects in the system. The fractional-order space-time Biswas-Arshed equation captures a variety of spatial and temporal dynamics including both linear and nonlinear interactions and local and nonlocal effects of fractional-order derivatives. These equations can be solved numerically or by sophisticated mathematical techniques, and the results have applications in physics, engineering, and mathematical modeling of real-world phenomena.

The space-time fractional Biswas-Arshed equation was investigated for closed-form solutions through the $(G'/G)$-expansion method (Akbar et al., 2012), the extended sinh-Gordon equation expansion method (Bilal et al., 2022), the extended simplest equation method (Zayed and Shohib, 2019), the modified extended auxiliary equation mapping method (Akbulut and Islam, 2022), and the modified rational sine-cosine method (Ganie et al., 2024).

## Space-Time Fractional Sasa-Satsuma Equation

The Sasa-Satsuma equation is a nonlinear evolution equation that describes the propagation of optical pulses in certain nonlinear media using fraction-order space-time beta derivatives. In terms of the fraction-order of space-time beta derivatives, it is represented as:

$$i D_t^{\alpha}\psi + a D_x^{2\alpha}\psi + b\vee\psi ¿^2\psi + i[\delta D_x^{3\alpha}\psi + \beta\vee\psi ¿^2 D_x^{\alpha}\psi + \theta D_x^{\alpha}(\vee\psi ¿^2\psi)] = 0.$$

Here, the term $i D_t^{\alpha}\psi$ has the same meaning as discussed in Schrödinger Kerr law equation. The term $a D_x^{2\alpha}\psi$ involves spatial derivatives. In this case, it represents the second-order spatial derivative of $\psi$, where $\alpha$ is an non-integer. It describes the spatial dispersion of the wave function, indicating how the curvature of the wave function evolves in space. The term $b\vee\psi ¿^2\psi$ involves nonlinear interactions. Here, $¿\psi¿^2$ represents the squared magnitude of the wave function, indicating the intensity. This term describes how the intensity of the wave function interacts with itself leading to phenomena like self-focusing or self-defocusing. Terms like $i\delta D_x^{3\alpha}\psi$ and $\beta\vee\psi ¿^2 D_x^{\alpha}\psi$ involve higher-order spatial derivatives. In this case, they describe third-order spatial derivatives of the wave function. These terms capture higher-order dispersion effects and additional nonlinear interactions that may arise due to the spatial curvature and intensity distribution of the wave function. The term $\theta D_x^{\alpha}(\vee\psi ¿^2\psi)$ involves interactions between the intensity distribution and the spatial curvature of the wave function. It captures how changes in intensity affect the spatial evolution of the wave function and vice versa. $a, b, \delta, \beta$, and $\theta$ are parameters that characterize the properties of the medium.

When discussing the fractional-order space-time beta derivative within the Sasa-Satsuma equation, we observe that the equation reduces to a standard form involving standard derivatives in time and space. This representation simplifies the understanding of pulse propagation in nonlinear media under fractional-order differential operators. However, it's worth noting that the fractional derivative aspects become relevant when dealing with non-integer orders providing a more comprehensive description of wave propagation in complex media.

The space-time fractional Sasa-Satsuma equation has been studied for finding exact analytic solutions by the F-expansion method (Li et al., 2020), the inverse scattering transform method (Ablowitz et al., 1974), the extended F-expansion method (Borg et al., 2024), and the Riemann-Hilbert method (Wu, and Geng, 2017).

## Fractional Space-Time Nonlinear Kaup-Newell Equation

The fractional space-time nonlinear Kaup-Newell equation is given by:

$$i D_t^{\alpha}\psi + a D_x^{2\alpha}\psi + ib D_x^{\alpha}(\vee\psi ¿^2\psi) - ic\psi D_x^{\alpha}(\vee\psi ¿^2) = 0.$$

The discussion about the above-mentioned equation is as follows: The terms $i D_t^{\alpha}\psi$ and $a D_x^{2\alpha}\psi$ have the same meaning as discussed in Schrödinger Kerr law equation. The term $ib D_x^{\alpha}(\vee\psi ¿^2\psi)$ have the same meaning as discussed in Schrödinger-Hirota equation. Here, $¿\psi¿^2$ represents the intensity of $\psi$, and $b$ is a coefficient governing the strength of nonlinearity. The derivative $D_x^{\alpha}$ with fractional-order $\alpha$ captures the spatial distribution of the intensity of the wave function. This term represents how the intensity-dependent nonlinearity affects the spatial evolution of the wave function. The term $-ic\psi D_x^{\alpha}(\vee\psi ¿^2)$ represent the nonlinear interaction term involving the wave function and the fractional spatial derivative of its intensity. In the context of the fractional space-time nonlinear Kaup-Newell equation, this equation, which takes into account both temporal and spatial non-local effects with integer-order fractional derivatives, describes the evolution of the wave function $\psi$. It gives a description of wave propagation in nonlinear media with fractional derivatives by incorporating nonlinear interactions and spatial dispersion.

The space-time fractional Kaup-Newell equation was taken into account for constructing their solutions through different methods like the modified Kudryashov method and the extended tanh function method (Wang, 2024), the new mapping method (Mahmood et al., 2024), the Hirota bilinear method (Ma et al., 2022), the new Kudryashov's method (Esen et al., 2022), and the extended simplest equation method (Ahmed et al., 2020).

## (2+1)-Dimensional Extended Time-Fractional Zakharov-Kuznetsov (ZK) Equation

The ion-acoustic model, in the context of the (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov (ZK) equation, can be expressed as:

$$D_t^{\alpha}\psi + a(\psi\psi_x) + b(\psi_{xxx} + \psi_{yyy}) + c(\psi_{xyy} + \psi_{xxy}) = 0.$$

Here, $\psi(x, y, t)$ represents the ion-acoustic wave potential, while the coefficients $a, b$, and $c$ are constants associated with the nonlinear interaction and dispersion in two spatial dimensions. The term $D_t^{\alpha}\psi$ represents the time-fractional derivatives which capture the non-local behaviour of ion-acoustic waves and provide a more accurate description of wave propagation in complex plasma environments. In ion-acoustic models, it accounts for the time-dependent behaviour of these waves and introduces a non-integer order temporal evolution that better describes anomalous diffusion or dissipation in the plasma. The parameter $\alpha$ (with $0<\alpha<1$) controls the degree of this fractional temporal evolution. This term $a(\psi\psi_x)$ is a nonlinear term involving the product of the wave function $\psi$ and its spatial derivative $\psi_x$ (the partial derivative of $\psi$ with respect to the spatial variable $x$). This term models the interaction between the wave function and its gradient, which can lead to wave steepening and the formation of solitons in the ion acoustic waves. The term

$b(\psi_{xxx}+\psi_{yyy})$ includes third-order spatial derivatives of the function $\psi$, where $\psi_{xxx}$ and $\psi_{yyy}$ are the third derivatives of $\psi$ with respect to $x$ and $y$ respectively. They describe the dispersive properties of ion-acoustic waves, allowing for the formation of smooth wave structures and balancing the nonlinearity. $c(\psi_{xyy}+\psi_{xxy})$ is a mixed derivative term. The mixed derivatives $\psi_{xyy}$ and $\psi_{xxy}$ describe the coupling between the spatial dimensions $x$ and $y$. These terms account for the cross-dispersive interactions in the wave dynamics. Physically, it models the multi-dimensional effects of ion-acoustic waves in plasma, where interactions occur in more than one spatial direction simultaneously.

The (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov equation was investigated for exact analytical solutions by several methods, for example the group analysis method (Liu et al., 2020), the variational iteration method (Zahran et al., 2021), and the sine-Gordon expansion method (Baskonus et al., 2017). San and Surgin (2023), Mabenga et al. (2023), employed the sub equation and classical symmetry methods to investigate the exact solitary wave, refracted wave and periodic wave solutions to the (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov model.

## Summary of Soliton Solutions Found

So, found solutions like V-shaped, positive-periodic, anti-asymptotic, breather, kink-bell-shaped, general solutions, asymptotic, periodic, kink, bell-shaped, anti-kink, periodic-type solitons, and others, which are important for analyzing the physical behavior of complex mathematical models, the nonlinear fractional evolution equation, and the nonlinear wave equation. These solutions provide insights into different types of wave propagation, stability, and interactions, which are essential for constructing a coherent or complementary model. These solutions are not only mathematically rich but also provide physical interpretations that are key to the application of these models in the real world, such as optical fibers, fluid dynamics, and quantum mechanics.

We have used beta derivative and the double $(\phi'/\phi, 1/\phi)$-expansion method in the space-time fractional (2+1)-dimensional Nizhnik-Novikov-Veselov, nonlinear Schrödinger Kerr law, unstable nonlinear Schrödinger, Schrödinger-Hirota, Biswas-Arshed, Sasa-Satsuma, nonlinear Kaup-Newell, (2+1)-dimensional extended time-fractional Zakharov-Kuznetsov models and have gotten asymptotic, anti-asymptotic, periodic, Smooth periodic, plane type, parabolic, breathing, irregular periodic, breathing-like periodic, rhythmic-breather, positive periodic shape, V-shaped, homoclinic-breather, heteroclinic-breather, multi-soliton shape, lump-periodic, breather-lump periodic, breather-periodic and general soliton solutions.

Fractional-order derivatives offer greater flexibility compared to integer-order derivatives. Modification of fractional order allows for fine-tuning the model's behavior to better align with empirical data or theoretical predictions. Utilizing fractional-order derivatives proves to be the effective approach in capturing the behaviors observed in various natural phenomena, such as complex biological systems, anomalous diffusion, and viscoelastic materials. These phenomena can be more accurately represented through a comparison between fractional-order and integer-order derivatives. Fractional-order derivatives provide a smoother transition between integer-order derivatives. For example, a fractional-order derivative of 1.5 lies between the first and second derivatives and accurately represents intermediate behaviour. In fields like signal processing, the fractional-order derivative is a valuable tool for managing signals with varying degrees of smoothness. Some systems are inherently fractional-dynamic which means that fractional-order differential equations are the better way to model them. In some situations, using fractional-order

derivatives is both desirable and necessary for accurately simulating the system. Fractional-order derivatives capture memory effects and non-local behaviours that integer-order derivatives cannot. This is particularly useful in systems where past states or events influence current behaviour, such as in viscoelastic materials, anomalous diffusion, or biological systems. Fractional-order derivatives can describe systems with smoother or less regular dynamics compared to integer-order derivatives.

The double $(\phi'/\phi, 1/\phi)$-expansion method is significant because it simplifies and solves complex differential equations, provides insights into solution behaviour, and is applicable across various scientific and engineering disciplines where other methods might be less effective.

## Methods and Importance of Exact Solutions

Exact solutions are important for validating methods, gaining analytical insights, simplifying problems, benchmarking results, and applying theoretical knowledge to real-world situations. Exact solutions can reveal critical insights into the dynamics of the system, such as wave propagation, stability, and interactions. In practical applications, exact solutions can provide direct answers and actionable insights, simplifying complex problems.

Some methods for finding an exact solution are Darbox transformation, Hirota bilinear method, Modified simple equation method, Substitution Methods, Methods of Undetermined Coefficients, Integral Transform Methods, Methods of Integrating Factors, Methods of Characteristics, Transform Methods, Symmetry Methods, Modified Expansion Method, Inverse Scattering Transform Method, and so on.

## References


Abdalla et al., (2018). On the soliton solutions to the Nizhnik-Novikov-Veselov and the Drinfel'd-Sokolov systems via the extended sinh-Gordon equation expansion method. Opt. Quant. Electron., 50, 1-15.

Abdelrahman M. A. E., Ammar S. I., Abualnaja K. M., and Inc M., (2020). New solutions for the unstable nonlinear Schrödinger equation arising in natural science. AIMS Mathematics., 5(3), 1893.

Ablowitz M. J., Kaup D. J., Newell A. C., and Segur H., (1974). The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems. Studies in Applied Mathematics., LIII, 4.

Ahmed H. M., Rabie W. B., Arnous A. H., and Wazwaz A. M., (2020). Optical solitons in birefringent fibers of Kaup-Newell's equation with extended simplest equation method. Phys. Scr., 95, 115214.

Akbulut A., and Islam S. M. R., (2022). Study on the Biswas-Arshed Equation With the Beta Time Derivative. Int. J. Appl. Comput. Math., 8, 167.

Akbar M. A., Ali N. H. M., and Zayed E.M.E., (2012). A generalized and improved ($G'/G$)-expansion method for nonlinear evolution equations. Math. Probl. Eng., 20, 12-22.

Akbar M. A., Ali N. H. M., and Zayed, E. M. E., (2012). Abundant exact traveling wave solutions of the generalized Bretherton equation via ($G'/G$)-expansion method. Commun. Theor. Phys., 57, 173-178.

Akbar M. A., and Ali N. H. M., (2012). New solitary and periodic solutions to nonlinear evolution equation by Exp-function method. World. Appl. Sci. J., 17, 1603-1610.

Alain M., Fernand M. L., Enyegue A. N., and Bioule M., (2020). Exact solitary waves for the 2D Sasa-Satsuma equation. Chaos, Solitons Fract., 133, 109657.

Al-Shawba., Gepreel A.. Abdullah K., and Azmia F., (2018). Abundant closed form solutions of conformable time fractional Sawada-Kotera-Ito equation using $(G'/G)$-expansion method. Results Phys., 9, 337-343.

Alqarni A. A., Ebaid A., Alshaery A. A., Bakodah H. O., Biswas A., Khan S., Ekici M., Zhou Q., Moshokoa S. P., Belic M. R., (2019). Optical solitons for Lakshmanan-Porsezian-Daniel model by Riccati equation approach. Optik., 182, 922-929.

Aly R. S., Naila N., and Dian-Chen L., (2021). Optical soliton and elliptic functions solutions of Sasa-Satsuma dynamical equation and its applications. Appl. Math-A. J. Chin. Uni., 36, 229-242.

Arnous A. H., Mirzazadeh M., Zhou Q., Moshokoa S.P., Biswas A., and Belic M., (2016). Optical solitons with higher order dispersions in parabolic law medium by trial solution approach. Optik., 127; 11306-11310.

Arshad M., Seadawy A. R., and Lu D., (2017). Modulation stability and optical soliton solutions of nonlinear Schrodinger equation with higher order dispersion and nonlinear terms and its applications. Superlattices and Microstructures., 112, 422-434.

Arshad M., Seadawy A. R., Lu D., and Ali F., (2020). Solitary wave solutions of Kaup-Newell optical fiber model in mathematical physics and its modulation instability. Modern Phys. Lett. B., 34, 2050277.

Arshad M., Seadawy A. R., and Lu D., (2017a). Study of soliton solutions of higher-order nonlinear Schrödinger dynamical model with derivative non-Kerr nonlinear terms and modulation instability analysis, Results Phys., 13, 102305.

Arshad M., Seadawy A. R., and Lu D., (2017b). Exact bright-dark solitary wave solutions of the higher-order cubic-quintic nonlinear Schrödinger equation and its stability. Optik., 138, 40-49.

Arshad M., Lu D., and Wang J., (2017). $(N+1)$-dimensional fractional reduce d differential transform method for fractional order partial differential equations, Commun. Nonlin. Sci. Numer. Simul., 48, 509-519.

Arshad M., Seadawy A. R., Lu D., and Jun W., (2018). Optical soliton solutions of unstable nonlinear Schrödinger dynamical equation and stability analysis with applications. Optik., 157, 597-605.

Baskonus H. M., Bulut H., and Sulaiman T. A., (2017). Investigation of various travelling wave solutions to the extended (2+1)-dimensional quantum ZK equation. Eur. Phys. J. Plus., 132, 1-8.

Bilal M., Rehman S. U.. and Ahmad, J., (2022). Dynamics of Diverse Optical Solitary Wave Solutions to the Biswas-Arshed Equation in Nonlinear Optics. Int. J. Appl. Comput. Math., 8, 137.

Biswas A., Jawad A. J. M., Manrakhan W. N., Sarma A. K., and Khan K. R., (2012). Optical solitons and complexitons of the Schrodinger-Hirota equation. Optics ¿ Laser Technology., 44, 2265-2269.

Borg M., Badra N. M., Ahmed H. M., and Rabie W. B., (2024). Solitons behavior of Sasa-Satsuma equation in birefringent fibers with Kerr law nonlinearity using extended F-expansion method. Ain Shams Engr. J., 15, 102290.

Chen., and Wang., (2020). Application of Bell polynomial in the generalized (2+1)-dimensional Nizhnik-Novikov-Veselov equation. Nonlinear Dynamics., 111, 1235-1245.

Ciancio et al., (2022). On the complex mixed dark-bright wave distributions to some conformable nonlinear integrable models. Fractals., 30, 2240018.

Esen H., Secer A., Ozisik M., and Bayram M., (2022). Dark, bright and singular optical solutions of the Kaup-Newell model with two analytical integration schemes. Optik., 261, 169110.

Fong., (2015). A novel generalized Kudryashov method for exact solutions of nonlinear evolution equations. AIP Conference Proceedings., 1798, 020082.

Ganie A. H., Baidani M. M. A., Wazwaz A. M., Ma W. X., Shamima U., and Ullah M. S., (2024). Soliton dynamics and chaotic analysis of the Biswas-Arshed model. Opt. Quant. Electron., 56, 1379.

Ilyas et al., (2021). The First Integral Method and Optical Soliton Solutions. Optik., 243, 166752.

Inc M., Aliyu A. I., Yusuf A., and Baleanu D., (2018a). Optical and singular solitary waves to the PNLSE with third order dispersion in Kerr media via two integration approaches. Optik., 163, 142-151.

Khatun M.A., Arefin M. A., Akbar M. A., and Uddin M. H., (2024). Existence and uniqueness solution analysis of time-fractional unstable nonlinear Schrödinger equation. Results Phys., 57, 107363.

Li C., Chen L., and Li G., (2020). Optical solitons of space-time fractional Sasa-Satsuma equation by F-expansion method. Optik., 224, 165527.

Liu J., Duan M., Mu G., and Wang C., (2010). Abundant exact solutions for the higher order non-linear Schrödinger equation with cubic-quintic non-Kerr terms. Commun. Nonlinear. Sci. Numer. Simulat., 15, 3777-3781.

Liu J. G., Yang X. J., Feng Y. Y., and Cui P., (2020). On group analysis of the time fractional extended (2+1)-dimensional Zakharov-Kuznetsov equation in quantum magneto-plasmas. Math. Comput. Simul., 178, 407-421.

Lu D., Seadawy A. R., and Arshad M., (2018). Bright-dark solitary wave and elliptic function solutions of unstable nonlinear Schrödinger equation and their applications. Opt. Quant. Electron., 50, 23.

Ma H., Gao Y., and Deng A., (2022). Dynamical analysis of diversity lump solutions to the (2+1)-dimensional dissipative Ablowitz-Kaup-Newell-Segure equation. Commun. Theor. Phys., 74, 115003.

Mabenga C., Muatjetjeja B., and Motsumi T. G., (2023). Bright, dark, periodic soliton solutions and other analytical solutions of a time-dependent coefficient (2 + 1) -dimensional Zakharov-Kuznetsov equation. Opt. Quant. Electron., 55, 1117.

Mahmood A., Abbas M., Nazir T., Abdullah F. A., Alzaidi A. S. M., and Emadifar H., (2024). Optical soliton solutions to the coupled Kaup-Newell equation in birefringent fibers. Ain Shams Engr. J., 15, 102757.

Moosaeia H., Mirzazadeh M., and Yildirim A., (2011). Exact solutions to the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity by using the first integral method. Nonlinear Anal. Model. Control., 16(3), 332-339.

Montazeri S., Nazari F., and Rezazadeh H., (2024). Solitary and periodic wave solutions of the unstable nonlinear Schrödinger's equation. Optik., 297, 171573.

Onder I., Secer A., Ozisik M., and Bayram M., (2024). Retrieval of the optical soliton solutions of the perturbed Schrödinger-Hirota equation with generalized anti-cubic law nonlinearity having the spatio-temporal dispersion. Math. Meth. Appl. Sci., 1-15.

Ray S. S., (2020). Dispersive optical solitons of time-fractional Schrödinger-Hirota equation in nonlinear optical fibers. Physica A., 537. 122619.

San S., and Sargin S., (2023). Dynamical properties and novel wave solutions of the time-fractional extended (2+1)-dimensional Zakharov-Kuznetsov equation in plasma physics. Opt. Quant. Electron., 55, 741.

Sarwar A., Gang T., Arshad M., Ahmed I., and Ahmad M. O., (2023). Abundant solitary wave solutions for space-time fractional unstable nonlinear Schrödinger equations and their applications. Ain Shams Engr. J., 14(2), 101839.

Sharif A., (2023). Jacobi elliptic function approach to a conformable fractional nonlinear Schrödinger-Hirota equation. Partial. Diff. Eq. Appl. Math., 8, 100541.

Wang K. L., (2024). Novel solitary wave and periodic solutions for the nonlinear Kaup-Newell equation in optical fibers. Opt. Quant. Electron., 56, 514.

Wang C., and Dai Z., (2015). Dynamic behaviors of bright and dark rogue waves for the (2+1) dimensional Nizhnik-Novikov-Veselov equation. Phys. Scr., 90, 065205.

Wang and Liu., (2022). SITEM for the Conformable Space-Time fractional (2+1)-Dimensional Asymmetrical Nizhnik-Novikov-Veselov Equations. Mathematics., 10, 1234.

Wazwaz, A.M., (2008). A study on linear and nonlinear Schrödinger equations by the variational iteration method. Chaos, Solitons Fract., 37(4), 1136-1142.

Williams., (2017). Multiwave solutions of time-fractional (2+1)-dimensional Nizhnik-Novikov-Veselov equations. Pamana-J. Phys., 88, 67.

Wu J. P., and Geng X. G., (2017). Inverse Scattering Transform of the Coupled Sasa-Satsuma Equation by Riemann Hilbert Approach. Commun. Theor. Phys., 67, 527-534.

Wu., and He., (2020). Homogeneous Balance Method for Solitonic Structures. J. Appl. Math. Comput., 384, 125396.

Zahran E. H. M., Bekir A., and Ahmed H., (2021). A variety of exact solutions of the (2+1)-dimensional modified Zakharov-Kuznetsov equation. Modern Phys. Lett. B., 35, 33.

Zayed E. M. E., and Shohib R. M. A., (2019). Optical solitons and other solutions to Biswas-Arshed equation using the extended simplest equation method. Optik., 185, 626-635.

Zhang et al., (2020). Modified Riemann-Hilbert Method for Nonlinear Schrödinger Equations. J. Nonlinear Analysis: Real World Applications., 53, 103079.

Zhang et al., (2022). Nonlinear superposition between lump waves and others waves of the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation. Nonlinear Dynamics., 108, 555-508.

Zhang Z., Huang J., Zhong J., Dou S. S., Liu J., Peng D., and Gao T., (2014). The extended $(G'/G)$-expansion method and travelling wave solutions for the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity. Pramana-J. Phys., 82(6), 101-1029.

Zhang L., Lin Y., and Liu Y., (2014). New solitary wave solutions for two nonlinear evolution equations. Comput. Math. Appl., 67, 1595-1606.

Zhao and Zhang., (2023). Exploring stochastic dynamics with different wave structures for the Nizhnik-Novikov-Veselov system. Opt. Quant. Electron. 55, 1-15.